The development of the classical Chinese algebra of polynomials

From the *Nine chapters* and Liu Hui through Wang Xiaotong, Li Ye and Zhu Shijie

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Mathematical texts from the Song and Yuan periods use a remarkably sophisticated algebraic method, describing manipulations of what we would call 'polynomial expressions' to solve geometric problems. This was the highest point of a development that started more than a millennium before, with the *Nine chapters* (*Jiuzhang suanshu* 九章算术) and Liu Hui 刘徽 (3rd cent. CE). In this article I analyse examples from four mathematical books and show some high points in the development of the system.

The story starts with the square- and cube-root algorithms in the *Nine chapters* and Liu Hui's geometric explanation of them. These are essentially methods of numerical solution of the simple polynomial equations $x^2 = a$ and $x^3 = a$. One geometric problem in the book involves what amounts to the solution of a non-trivial quadratic equation, $x^2 + 34x = 71{,}000$. Wang Xiaotong in the 7th century uses *ad hoc* geometric constructions to create cubic equations that can be solved numerically. Li Ye in the 13th century manipulates polynomial expressions without reference to their geometric origin. Zhu Shijie in 1303 extends the system in a number of ways, the most important of which is that he can deal with polynomials of more than one variable. After this the system was completely forgotten until it was revived as a matter of historical interest by mathematicians in the 19th century.

Liu Hui and the *Nine Chapters*

The *Jiuzhang suanshu* (Arithmetic in nine chapters)[1] is certainly the most important book in the history of Chinese mathematics. Its date has long been uncertain. Zou Dahai (2022: 34–51) has reviewed many attempts to date the book, and concluded that its origin was before the Qin period, but reached its present form in the Western Han.[2] I

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[1] This title has been translated in several different ways. Since *jiuzhang* modifies *suanshu* a pedantically correct translation must have the form '... in nine chapters'. How to translate *suanshu* is a matter of translation philosophy: the word means 'arithmetic', but since the book contains much more than simple arithmetic it would be equally correct to translate the title as 'Mathematics in nine chapters' or similar.

[2] I reach the same conclusion, by a slightly different path. In considering any narrative historical source, one must consider the question of 'the source's source': How does this author know what he is telling us? The same question must be asked about his source, that source's source, and so forth. Liu Hui states in his preface that the book's origin was before the Qin 'burning of the books', 213 BCE, and that Zhang Cang 张苍 (1st cent. BCE) and Geng Shouchang 耿寿昌 (2nd cent. BCE) reconstituted and revised what was left of it. Liu Hui is likely to have had this information from an edition by one or both of these scholars, and they were in a position to know a good deal about the age of the material that they worked with. A very important point is that Liu Hui says the language of the book is in large part modern (*Suo lun zhe duo jin yu ye* 所论者多近语也). He was in a much better position than we are, 18 centuries later, to evaluate the language of the book.

feel no need to go much further into this question here: the book is early. Liu Hui's commentary, which can be securely dated to 263 CE, corrects, extends, and explains its methods.[3]

The algorithms for extracting square and cube roots are in chapter 4.[4] In explaining the square root algorithm Liu Hui uses a diagram, now lost, that may have resembled Figure 1d.

Problem 12 in chapter 4 asks for the square root of 55,225. The *Jiuzhang* text describes the root extraction step by step, and from its terminology it is clear that this algorithm had from the beginning been interpreted geometrically. Liu Hui makes the geometric interpretation explicit.

See Figures 1a–d. The starting situation is a square with area 55,225 square paces. The first digit of the root, 2, is determined. Its value is 200 paces; a corresponding square is cut out of the original square, leaving a gnomon with area 15,225 square paces. The second digit is 3, and a gnomon corresponding to 30 paces is removed, leaving 2,325 square paces. The third digit is is 5, and when the corresponding gnomon is removed there is nothing left of the original square. The result is therefore exactly 235 paces.

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[3] The book and the commentary have been translated into numerous languages, including Japanese and modern Chinese. I believe the best translation to a Western European language is the French translation by Karine Chemla and Guo Shuchun (2004). English translations include Guo Shuchun et al. (2013) and Shen Kangsheng et al. (1999).

[4] *Shao guang* 少广 ('lesser widths', an enigmatic title), a chapter largely about the calculations of areas and volumes. Qian Baocong 1963: 149–151; Chemla & Guo 2004: 322ff., 360–368, 800–804 n. 27–55; Guo Shuchun et al. 2013: 373–409; Shen Kangsheng et al. 1999: 175ff., 203–213.

The *Jiuzhang suanshu* includes in a single case the direct use of the geometric interpretation of the square-root algorithm to solve a geometric problem, problem 20 in chapter 9.[5] This and Liu Hui's comment are translated in Appendix 1.

It is quite possible that Liu Hui provided a diagram, now lost, with his comment, but Figure 2 is simply my diagram, not an attempt to reconstruct his. A square city, *LKPS*, has gates at *B* and *C*, the centres of *LK* and *SP*. A tree is at *A*. Someone walks from *C* to *D* and then to *E*, where he sees the tree. The given quantities are

a = 20 paces

b = 14 paces

c = 1,775 paces

and x is to be calculated. Liu Hui uses the similarity of triangles *ABL* and *ADE* to show that $2ac$ is equal to the area of the rectangle *FGHJ*. He then dissects and rearranges areas to construct the gnomon *LGQPRU* in Figure 3, which has the same area and is essentially the same as the result of the first step of the square-root algorithm (Figure 1b). Here ½(a+b) corresponds to the first digit of a square root. Continuing the steps of the algorithm gives x = 250 paces. In a modern interpretation this corresponds to solving the polynomial

$$x^2 + (a + b)x = 2ac$$

$$x^2 + 34x = 71{,}000 \text{ square paces}$$

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[5] Qian Baocong 1963: 255–256; Chemla & Guo 2004: 689–693, 736–738; 892–893 n. 107–112; Guo Shuchun et al. 2013: 1137–1143; Shen Kangsheng et al. 1999: 507–512.

This is then the first known use of a non-trivial polynomial in classical Chinese mathematics, the beginning of a long development. It is interpreted entirely geometrically, without the use of any algebraic reasoning.

Wang Xiaotong's *Continuation*

Wang Xiaotong 王孝通 was an official of the Sui and Tang dynasties concerned with mathematics education, astronomy, and the calendar. His dates are tentatively given as '579? – 638?'. His book, *Jigu suanjing* 緝古算經, 'The continuation of ancient mathematics', was presented to the Tang Emperor around 638 (Lim & Wagner 2017: 3–5).

In this book the numerical solution of cubic polynomials by the Chinese version of Horner's Method[6] is part of the background knowledge assumed on the part of the reader. This root extraction is directly related to the extraction of square and cube roots in the *Nine chapters* (and other early books), as is clear from the terminology used.

Reading the book it becomes clear that Wang Xiaotong sees the columns of numbers used in the root-extraction algorithm as representing what we would call equations: they are statements that certain operations on an unknown quantity result in a certain quantity.

Building a tamped-earth dyke

Two of the problems in Wang Xiaotong's book are translated in Appendices 2 and 3. In the first, the second part of Problem 3, the dimensions and volume are given for a tamped-earth dyke (Figure 4) to be built by workers from four counties. The volume that each county is responsible for is given, and the dimensions of the county's part are to be calculated. Many of the book's problems are 'textbook exercises' without direct

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[6] See e.g. Wagner 2017.

practical application,[7] but this one is real, a simplified version of one of the tasks of Imperial officials: administering the assignment of corvée labour for public works (Wagner 2013; Lim & Wagner 2017: 6–12).

See Figure 4. The dimensions of the whole dyke have been determined in the first part of Problem 3:

a = 80 *cun* 寸

b_1 = 142 *cun*

b_2 = 762 *cun*

h_1 = 31 *cun*

h_2 = 341 *cun*

l = 4,800 *cun*

The volume of County A's section is

V_A = 33,351,040 cun^3

In the second part of the problem the length of County A's section, x, is determined, after which its other dimensions are determined by considering similar triangles. A cubic equation is built up using the volume dissection shown in Figure 5. The volumes of the four parts are all products of known quantities and powers of x:

$$V_{\mathrm{I}} = \frac{1}{2}(a + b_1)h_1 x = 3{,}441x$$

[7] For example, the first part of this same Problem 3 gives the total volume of the dyke and the *differences* between various dimensions; the dimensions are to be calculated. This is clearly not a computation encountered in real public works.

$$V_{\mathrm{II}} = \tfrac{1}{2}\,\frac{x}{l}(h_2 - h_1)b_1 x = \frac{b_1(h_2 - h_1)}{2l}x^2 = \frac{2{,}201}{480}x^2$$

$$V_{\mathrm{III}} = 2 \times \tfrac{1}{6}\frac{x}{l}\left(\frac{b_2 - b_1}{2}\right)\frac{x}{l}(h_2 - h_1)\,x = \frac{(b_2 - b_1)(h_2 - h_1)}{6l^2}x^3 = \frac{961}{691{,}200}x^3$$

$$V_{\mathrm{III}} + V_{\mathrm{II}} + V_{\mathrm{I}} = V_{\mathrm{A}}$$

$$\frac{(b_2 - b_1)(h_2 - h_1)}{6l^2}x^3 + \frac{b_1(h_2 - h_1)}{2l}x^2 + \tfrac{1}{2}(a + b_1)h_1 x = V_{\mathrm{A}}$$

$$\frac{961}{691{,}200}x^3 + \frac{2{,}201}{480}x^2 + 3{,}441x = 33{,}351{,}040\ cun^3$$

However, Wang Xiaotong’s cubics always have the cubic coefficient equal to 1. His calculation is therefore

$$x^3 + \frac{3b_1 l}{b_2 - b_1}x^2 + \frac{3(b_2 - b_1)(h_2 - h_1)}{(a + b_1)h_1}l^2 x = \frac{6l^2}{(b_2 - b_1)(h_2 - h_1)}V_{\mathrm{A}}$$

$$x^3 + 3{,}298\tfrac{2}{31}\,x^2 + 2{,}474{,}941\tfrac{29}{31}\,x = 23{,}987{,}761{,}548\tfrac{12}{31}\,cun^3$$

This equation has one real root,

$$x = 1{,}920\ cun$$

A triangle problem

At the end of Wang Xiaotong’s book are six problems in plane geometry which he solves using three-dimensional constructions. Two of these, Problems 19–20, are translated in Appendix 3 below. A challenge here is that all extant versions of the *Continuation* go back to a damaged copy (now lost) in which large parts of the last pages are fragmentary. Various attempts have been made to reconstruct the original text. These are discussed by Lim and Wagner (2017: 104–108); for convenience I accept here Zhang Dunren's 張敦仁 reconstruction. Others differ in detail but not in the overall method which is my concern here.

Problem 19 states that in a right triangle (see Figure 6),

$$a = 7\tfrac{7}{10}$$

$$bc = 726$$

and *b* is to be determined. No units are given for these quantities. Wang Xiaotong's six triangle problems may be the only place in the entire classical Chinese mathematical corpus in which geometric quantities are stated without units.

The calculation is then the numerical solution of

$$x^2 + a^2x = (bc)^2 \qquad (1)$$

$$x^2 + 59\tfrac{20}{100}x = 52{,}706$$

in which $x = b^2$. This has one positive root,

$$x = 696\tfrac{24}{25}$$

and

$$b = \sqrt{x} = 26\tfrac{2}{5}$$

It is easy to derive (1) algebraically, for

$$b^4 + a^2b^2 = b^2(a^2 + b^2) = b^2c^2 = (bc)^2$$

However, what remains of the fragmentary text of a comment in smaller characters appears to imply a geometric explanation. That in turn would seem to imply a four-dimensional construction. But the lack of units in the given quantities makes the construction shown in Figure 7 plausible. Here both *b* and b^2 are used as linear measures. The volume of the solid is b^2c^2, and the sum of the two blocks into which the solid is divided is

$$b^4 + (c^2 - b^2)b^2 = x^2 + a^2x = (bc)^2$$

These two examples show that by Wang Xiaotong's time an important insight had been gained: the columns of numbers used in root extraction are what we call polynomial equations, and they can be used in more complex calculations than were dealt with in the *Nine chapters*. He does not manipulate these, but constructs them using *ad hoc* geometric constructions. His algebra is thus still tightly tied to geometry.

Li Ye's *Sea mirror*

Ceyuan haijing 測圓海鏡 (Sea mirror of circle measurements), by Li Ye 李冶 (1192–1279) is one of several books of the late Song and Yuan periods that use manipulations of polynomials in solving complex mathematical problems (Mei Rongzhao 1966; Martzloff 1997: 17–18). The system was called *Tianyuan shu* 天元術, 'the method of the celestial unknown'.[8] The variable in the polynomial was called the *tianyuan*.

The *Sea mirror* appears to be the one of these books that uses the method in its most general form, at least with respect to polynomials of one variable. Later Zhu Shijie 朱世傑 would extend it to work with up to four unknowns.

The book begins with a diagram, reproduced here in Figure 8, which shows a circular city and a number of roads inside and outside the city. Here Chinese characters are used to mark points. In the same figure I give a corresponding modern diagram in which capital letters are substituted for the characters.

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[8] On the translation of *yuan* 元 as 'unknown' see fn. 10 below.

The diagram is followed by a chapter which states 692 geometric identities. These are the raw material for the core of the book: 171 problems demonstrating the use of the method of the celestial unknown. The parts of this chapter that are relevant for the following discussion are translated in Appendix 4.1.

One of the simpler problems, Problem 12 in Chapter 5, is translated in Appendix 4.2: A man walks from *Q* to *C* and then to *F* in Figure 8; he has walked a total of 1,144 paces. Another man walks from *G* to *T* and then to *F*; the difference between these two walks is 56 paces. The diameter of the circular city is required. Thus

$$QC + CF = 1{,}144 \text{ paces} \tag{2}$$

$$FT - TG = 56 \text{ paces} \tag{3}$$

and *d* is required. Then follows one of Li Ye's 171 demonstrations of the manipulation of polynomials to solve non-trivial problems. While this is obviously not a real-world problem, the method was definitely used in solving real practical problems in ancient China (see e.g. Wagner 2013).

Two non-obvious facts, stated in Chapter 1 (see Appendix 4.1.2), are used:

$$QC - CF = FT - TG \tag{4}$$

$$CK \times TG = \tfrac{1}{2}d^2 \tag{5}$$

These follow from two facts stated in the *Nine chapters* and proved by Liu Hui: [9]

$$d = QC + QT - CT = \frac{2 \times QC \times QT}{QC + QT + CT}$$

From these follow

[9] Qian Baocong 1963: 252–253; Chemla & Guo 2004: 726, 728–729; Guo Shuchun et al. 2013: 1110–1121; Shen Kangsheng et al. 1999: 494–495; Wagner 2022.

$$QC = CT - QT + d = (CF + FT) - (TG + d) + d = FT - TG + CF$$

$$QC - CF = FT - TG$$

and

$$QC \times QT = \tfrac{1}{2}d(QC + QT + CT)$$

$$CK \times TG = (QC - d)(QT - d)$$

$$= QC \times QT - d(QC + QT) + d^2$$

$$= \tfrac{1}{2}d(QC + QT + CT) - d(QC + QT) + d^2$$

$$= \tfrac{1}{2}d \times QC + \tfrac{1}{2}d \times QT + \tfrac{1}{2}d \times CT - d \times QC - d \times QT + d^2$$

$$= -\tfrac{1}{2}d(QC + QT - CT) + d^2$$

$$= -\tfrac{1}{2}d^2 + d^2 = \tfrac{1}{2}d^2$$

It follows immediately from (2), (3), and (4) that

QC = 600 paces

CF = 544 paces

The polynomials are, as before, columns of counting-rod numbers. Representations of these are included directly in the text, as can be seen in Appendix 4.2. How these small drawings are to be interpreted is shown in Figures 9 and 10.

Li Ye's text gives first the calculation of the coefficients of the cubic equation to be solved numerically:

$$0.5d^3 - 1{,}200d^2 + 427{,}200d = 40{,}320{,}000 \text{ cubic paces} \quad (6)$$

and gives the solution, 240 paces. Then follows the detailed derivation of this equation.

The first polynomial on the counting board is an expression for *CO*:

$$CO = QC - 0.5d = -0.5d + 600 \tag{7}$$

Next an expression for *FT* is needed. Considering similar triangles, this is

$$FT = \frac{0.5d \times CF}{CO}$$

In the following steps the divisor *CO* is 'set aside', so that the next polynomial is in effect

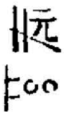
$$FT \times CO = 0.5d \times CF = 272d$$

Two further steps lead to an expression for *TG*:

$$(FT - TG) \times CO = 56 \times (-0.5d + 600) = -28d + 33{,}600$$

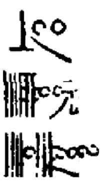
$$TG \times CO = (FT - [FT - TG]) \times \mathrm{CO} = 272d - (-28d + 33{,}600)$$
$$= 300d - 33{,}600$$

Now an expression for *CK* is needed, and the next polynomial is



$$2 \times CK = 2(QC - d) = -2d + 2 \times QC = -2d + 1{,}200$$

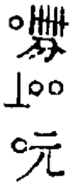
Multiplying this by the expression found for *TG* and using (5) gives an expression for d^2:



$$d^2 \times \mathrm{CO} = -QC^2 \times d^2 + (QC^2 + 2 \times 56 \times QC)d - 2 \times 56 \times QC^2$$

$$= -600d^2 + 427{,}200d - 40{,}320{,}000 \tag{8}$$

Now another expression for d^2 is needed. Using (7),



$$d^2 \times CO = 0.5d^3 + 600d^2 \quad (9)$$

Subtracting (8) from (9) gives

$$0.5d^3 - 2 \times QC \times d^2 + QC\,(2 \times 56 + QC)d - 2 \times 56 \times QC^2$$
$$= 0.5d^3 - 1200d^2 + 427{,}200d - 40{,}320{,}000 = 0$$

which is equivalent to (6).

Confirming the problem solutions. The equation (6) has three roots, $d = 168.737$, 240, and 1991.26. The third is clearly not a solution: $d > QC$. The first two are.

If QC and d are given, all of the other dimensions of the triangle are determined. From the Pythagorean Theorem and (4) can be derived:

$$QT = \frac{d(2 \times QC - d)}{2(QC - d)}$$

$$CT = \frac{2 \times QC^2 - 2d \times QC + d^2}{2(QC - d)}$$

$$TG = QT - d$$

and considering similar triangles,

$$FT = \frac{d \times CT}{2 \times QC}$$

$$CF = CT - FT$$

Then given *QC* = 600 it can be calculated that both *d* = 168.737 and *d* = 240 can satisfy (2) and (3).

This example shows us the classical Chinese algebra of polynomials in one variable close to its full development, used in solving complex problems. The system clearly has limitations. In particular, Li Ye states all of his problems in geometric terms, the unknown quantity is in each case a linear measure, and all of his polynomials are functions of this unknown.

Zhu Shijie's *Jade mirror*

In the *Siyuan yujian* 四元玉鉴 (Jade mirror of four unknowns[10], published in 1303), we see the classical Chinese algebra of polynomials at its highest level of sophistication. Here the columns of numbers representing polynomials in one unknown that we have seen in Li Ye's *Sea mirror* are extended to rectangular arrays of numbers which represent polynomials in up to four unknowns. The author, Zhu Shijie 朱世傑, appears to have drawn on several earlier works, but these are no longer extant (Du Shiran 1966: 168).

In Li Ye's polynomials the single unknown is called *tianyuan yi* 天元一 or simply *tianyuan*, 'celestial unknown'. In Zhu Shijie's book the unknowns are called *tianyuan* 天元, *diyuan* 地元, *renyuan* 人元, and *wuyuan* 物元, the 'celestial',

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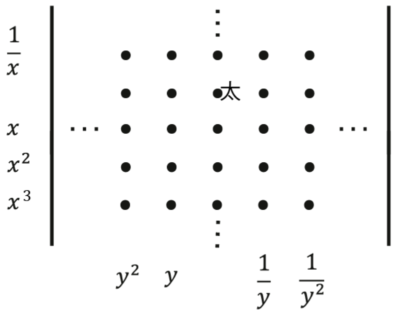
[10] The word *yuan* 元 in this title has been translated in several different ways. The earliest Western scholar to discuss the book, L. Vanhée (1931), uses *éléments*, and several scholars have followed him; Martzloff (1997: 17) uses the ordinary meaning of the word, *origin*. Hoe (1977; 2007) follows the mathematical meaning of the word in this particular context, using *inconnues* in French and *unknowns* in English; Guo Shuchun et al. (2006) do the same.

‘terrestrial’, ‘human’, and ‘creature’ unknowns. Modern scholars studying the *Jade mirror* represent these with *x*, *y*, *z*, and *u*.

In this book Zhu Shijie introduces his methods with four worked examples, using one, two, three, and four unknowns respectively. These have been translated several times.[11] This introduction is followed by statements of 284 problems with the barest of hints as to how they are to be solved. Fortunately the Qing scholar Luo Shilin 羅士琳 (d. 1853) in his *Siyuan yujian xicao* 四元玉鑑細草, published in 1837, gives detailed working (*xicao*) for each of these, and I depend heavily on his work in my discussion.

The counting-rod array notation for a polynomial in two variables is a rectangular array with a place for each coefficient, like this:



So that for example this array:

7	0	0	0	0
	6	5 太	0	0
			4	0
				3
				2

would represent the polynomial

$$\frac{2x^3}{y^2}+\frac{3x^2}{y^2}+\frac{4x}{y}+5+6y+\frac{7y^2}{x}$$

The use of counting-rods is in general the same as laid out in Figures 9 and 10. In illustrations of polynomials in the book the constant term is often but not always

[11] Vanhée 1931; Hoe 1977; 2007; Guo Shuchun et al. 2006: 42–85.

marked with the character *tai* 太, 'supreme'. The calculator might have marked this on the counting board in some way, or he might simply have kept the position in mind.

This notation appears to be perfectly general for polynomials in two unknowns. Extending it to a general notation for polynomials in three or four unknowns, however, would seem to require three- or four-dimensional arrays. In these cases Zhu Shijie uses a complex notation which I do not understand deeply enough to explain properly, and in the following I shall be concerned only with the case of two unknowns.

It is apparent that the array notation owes much to the matrix approach to systems of linear equations presented in the *Nine chapters*, chapter 8.[12] The terms *xiao* 消, 'clear', and *qi* 齊, 'homogenize', are used here in ways analogous to their use there.

Multiplying or dividing an array by x is done simply by moving the whole array down or up. The same operations with y move the array to the left or right. This moving of the array requires only moving the character *tai* 太 (or other indication of the constant term) accordingly.

The algorithm for multiplication of two arrays is never stated, but its result is of course the array corresponding to the product of the polynomials represented by the multiplicand arrays.

Each of the 288 problems in the book gives two equations, the first prefaced by *jin you* 今有, 'now there is ...', and the second by *zhi yun* 只云, 'it is only stated that . . .'. In Zhu Shijie's four worked examples two equations are developed that correspond to these, called the *jin* array and the *yun* array. From these are developed the 'left' and 'right' arrays, corresponding to two polynomials which are both equal to zero.

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[12] *Fangcheng* 方程, 'rectangular arrays'. Qian Baocong 1963: 221–240; Chemla & Guo 2004: 599–659, 861–877; Guo Shuchun et al. 2013: 905–1033; Shen Kangsheng et al. 1999: 386–438. The method presented in this chapter is essentially Gaussian elimination in a matrix representing a system of linear equations in several unknowns.

These are analogous to the matrices in the linear algebra of the *Nine chapters*; the operations of 'homogenizing' and 'clearing' are used to eliminate all unknowns except *x*. The Chinese version of Horner's method is then used to find a numerical root of this equation.

With only two equations given in each problem, how are problems of three or four unknowns solved? Most or all of these involve right triangles, and an implicit third equation is the Pythagorean theorem. And the worked example using four unknowns has only three actual unknowns, the three sides of a right triangle.

The example of a problem in two unknowns that I have chosen for this discussion is problem 15 of section 6.[13] It is translated, together with Luo Shilin's detailed working, in Appendix 5.

Letting *a* and *b* be the dimensions of a rectangle, the statements of Problem 15 amount, in modern notation, to

$$\sqrt{ab} + \sqrt{a+b} - a = 8 \text{ paces} \tag{10}$$

$$\sqrt{a+b} - \sqrt{b} = 1 \text{ pace} \tag{11}$$

Zhu Shijie directs the reader to let the two unknowns be

$$x = a, \quad y = \sqrt{a+b} \tag{12}$$

[13] The title of section 6 is *Zuo you feng yuan* 左右逢元, 'Both left and right one meets unknowns', a playful pun on a sentence in *Mencius*, *Zuo you feng qi yuan* 左右逢其原, 'wherever he looks [a gentleman] meets the sources [of the Way]' (*Mengzi* 孟子, book 4b, no. 14, Yang Bojun 1960, 1: 189).

and ‘calculate by combining’ these. He then gives the numerical coefficients of the final polynomial equation to be solved,

$$x^4 - 4x^3 - 26x^2 - 196x + 225 = 0 \tag{13}$$

together with the result $a = 9$ paces, $b = 16$ paces. He does not indicate how these coefficients are calculated, leaving to the reader the task of getting from (10) and (11) to (13).

Fortunately Luo Shilin gives all the details. Using modern notation, from (11),

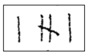
$$b = \left(\sqrt{a+b} - 1\right)^2 = (y-1)^2 = y^2 - 2y + 1 \tag{14}$$

Further, from (10),

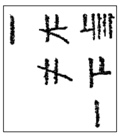
$$ab = \left(a - \sqrt{a+b} + 8\right)^2 = (x - y + 8)^2$$

$$= x^2 + 16x - 2xy - 16y + y^2 + 64 \tag{15}$$

Luo Shilin expresses these polynomials as arrays:

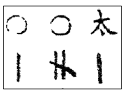


$$\begin{vmatrix} 1 & -2 & 1 \end{vmatrix} \qquad [b = y^2 - 2y + 1] \qquad (14')$$



$$\begin{vmatrix} 1 & -16 & 64 \\ & -2 & 16 \\ & & 1 \end{vmatrix} \qquad [x^2 + 16x - 2xy - 16y + y^2 + 64] \qquad (15')$$

Multiplying (14′) by x gives

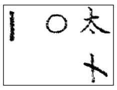


$$\begin{vmatrix} 0 & 0 & \text{太} \\ 1 & -2 & 1 \end{vmatrix} \qquad [ab = x - 2xy + xy^2] \qquad (16)$$

Subtracting (16) from (15′),

$$\begin{vmatrix} 1 & -16 & 64 \\ -1 & 0 & 15 \\ & & 1 \end{vmatrix} \qquad [x^2 + 15x - xy^2 - 16y + y^2 + 64 = 0] \qquad (17)$$

This he calls the *jin* 今 array. Further, from (11),



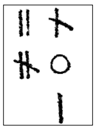
$$\begin{vmatrix} 1 & 0 & \text{太} \\ & & -1 \end{vmatrix} \qquad [b = -x + y^2] \qquad (18)$$

Subtracting (14′) from (18) gives

$$\begin{vmatrix} 2 & -1 \\ & -1 \end{vmatrix} \qquad [-x + 2y - 1 = 0] \qquad (19)$$

This he calls the *yun* 云 array.

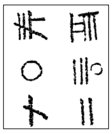
Luo Shilin has now arrived at two polynomials in *x* and *y*, (17) and (19), each of which equals zero. The next step is to eliminate the two y^2 terms in (17). Multiplying the *yun* array by the left column of the *jin* array, (19),



$$\begin{vmatrix} 2 & -1 \\ -2 & 0 \\ & 1 \end{vmatrix} \qquad [x^2 - 2xy + 2y - 1 = 0]$$

$$(-x + 2y - 1) \times (-xy + y) = x^2y - 2xy^2 + 2y^2 - y$$
$$= x^2 - 2xy + 2y - 1 = 0$$

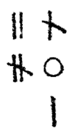
This is called the 'left' array. To make the 'right' array, the *jin* array, (17), is doubled and the 'left' array is subtracted from it,



$$\begin{vmatrix} -31 & 128 \\ 0 & 30 \\ -1 & 2 \end{vmatrix} \qquad [2x^2 - x^2y + 30x - 31y + 128 = 0]$$

$$2(x^2 + 15x - xy^2 - 16y + y^2 + 64) - (x^2 - 2xy + 2y - 1)$$
$$= 2x^2 - x^2y + 30x - 31y + 128 = 0$$

On the counting board are now the 'left' and 'right' arrays:



$$\begin{vmatrix} 2 & -1 \\ -2 & 0 \\ & 1 \end{vmatrix} \qquad \begin{vmatrix} -31 & 128 \\ 0 & 30 \\ -1 & 2 \end{vmatrix}$$

$$x^2 - 2xy + 2y - 1 = 0 \qquad 2x^2 - xy + 30x - 31y + 128 = 0 \qquad (20)$$

and the next step is to eliminate the four *y* terms. In this arrangement the product of the 'inner' columns is subtracted from the product of the 'outer' columns:

$$\begin{vmatrix} 225 \\ -196 \\ -26 \\ -4 \\ 1 \end{vmatrix} \qquad \begin{bmatrix} [-4x^3 - 56x^2 - 196x + 256] - [-x^4 - 30x^2 + 31] \\ = \\ x^4 - 4x^3 - 26x^2 - 196x + 225 = 0 \end{bmatrix} \tag{21}$$

$$\begin{aligned} &(-2xy + 2y) \times (2x^2 + 30x + 128) - (x^2 - 1) \times (-x^2y - 31y) \\ &\quad = x^4 - 4x^3 - 26x^2 - 196x + 225 = 0 \end{aligned}$$

This result is (13), the equation given by Zhu Shijie to be solved numerically. It has two real roots, $x = 1$ and $x = 9$; and $a = x = 9$ is the correct solution. Luo Shilin does not indicate how b is calculated, but one simple way is to substitute $a = 9$ paces into (10) and (11) and take their difference, giving the result $b = 16$ paces. The root $x = 1$ is not a solution, as no value of b can in that case validate both (10) and (11).

The calculation in (13)–(21) needs some explanation. It follows from the equations in (20) that

$$-2xy + 2y = -(x^2 - 1)$$

$$-xy - 31y = -(2x^2 + 30x + 128)$$

And the difference of the products of the inner and outer columns in (20) is

$$\begin{aligned} &(-2xy + 2y)(-2x^2 - 30x - 128) - (-x^2 + 1)(-xy - 31y) \\ &= (-x^2 + 1)(-2x^2 - 30x - 128) - (-x^2 + 1)(-2x^2 - 30x - 128) \\ &= 0 \end{aligned}$$

This example shows something of the sophistication that the algebra of polynomials reached in Zhu Shijie's book. Polynomials in more than one variable are allowed, the tie to geometry is entirely broken, and variables can point to functions of the unknown as in (10), $y = \sqrt{a + b}$.

Concluding remarks

In this article we have seen examples from four books ranging over more than a millennium. What can we say about the conceptual development that they show?

It is important to emphasize that more examples from more books would undoubtedly lead to more solid insights. Furthermore, there were many other mathematical books written and published in our period of interest that are no longer extant. And we must also assume that there were many mathematicians who contributed to this development but never wrote a book and whose names we do not know. Therefore, while we naturally admire the accomplishments of Liu Hui, Wang Xiaotong, Li Ye, and Zhu Shijie, we may not suppose that they alone moved the development forward.

The example from the *Nine chapters*, with Liu Hui's commentary, showed that a well-known technique could be used for more than it was designed for – this is often the first step in the evolution of a technique or a technology. But as long as the technique of root extraction was conceptually tied to geometry, problem 20 remained an interesting curiosity without practical application or obvious potential for further development.

The first major development that we see through these examples is the recognition in Wang Xiaotong's *Continuation* that the column of numbers used in the root-extraction process is what we would call an equation: it is in effect a statement that performing certain operations on an unknown quantity results in a certain quantity.

Wang Xiaotong forms his equations by *ad hoc* geometric constructions, so his algebra is still tied to geometry, but while Liu Hui saw the ‘coefficients’ of his ‘equation’ as areas or volumes, for Wang Xiaotong they are simply numbers that appear in the calculation of certain volumes.[14]

In the example from Li Ye’s *Sea mirror* we see a break in the connection to geometry. Though all of the problems in the book are stated in geometric terms, the actual calculations are entirely concerned with numbers.

Further, in this book a column of numbers is not always an equation to be solved numerically, but an *expression* for one unknown in terms of another. These expressions can be manipulated: added to, subtracted from, or multiplied by other expressions, or multiplied or divided by a constant. Looking only at the first steps in the example, the equation (7) states a relation between *CO* and *d*. The aim of the manipulations is to derive an expression in the unknown which is equal to zero. This is then an equation that can be solved numerically. The general method seems to be to find two different expressions in the unknown for the same variable and subtract the one from the other; in our example, equations (8) and (9), two expressions for $d^2 \times CO$.

In Zhu Shijie’s *Jade mirror* we see the highest level of sophistication that was reached in this algebraic system. It is extended to deal with up to four variables, manipulated using methods inspired by the matrix algebra of the *Nine chapters*.[15] An advance which has received less attention is greater freedom in the choice of variables. In Li Ye’s *Sea mirror* the variable in each polynomial expression is the unknown

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[14] While Wang Xiaotong sometimes names an intermediate quantity ‘area for’ some geometric figure, for example ‘Area for the End Wall’, called K_2 in Appendix 2, this quantity rarely corresponds to any actual area in the geometric situation. But calling the quantity an area indicates its dimensions.

[15] See fn. 12 above.

quantity of the respective problem. In the *Jade mirror* a variable can be a function; in our example, the variable $y = \sqrt{a+b}$ (see equation (12)). While Zhu Shijie has no general symbolism for a function like $\sqrt{a+b}$, and must state the calculation in words, he is able to deal with it by implicitly 'reasoning about calculations', as Wang Xiaotong before him had done in some cases (Lim and Wagner 2017: 51–52).

What were these books? Were they textbooks of practical mathematics? While their examples are often not at all practical, the mathematics that they present was definitely used in practical problem-solving.[16] Both the *Nine chapters* and Wang Xiaotong's *Continuation* were used in mathematical education in the Tang period (Lim and Wagner 2017: 13–20), and presumably later as well; but the *Sea mirror* and the *Jade mirror* seem not to have been used as textbooks. Perhaps they should be categorized as 'pure mathematics' or 'recreational mathematics' – though these are modern concepts, not necessarily applicable to ancient books.

Why was this marvellous algebra of polynomials totally forgotten by the late 14th century, the beginning of the Ming dynasty? No book later than Zhu Shijie's *Jade mirror* mentions it before the renewed interest of Chinese mathematicians in the 19th century. The introduction of the abacus around this time may be part of the explanation, together with revised administrative practices in public works. It may also be that 'pure mathematicians' had developed the system as far as it could be taken without a radical paradigm change, and therefore turned to other interests.

[16] Lim and Wagner 2017: 12; Wagner 2013; 2023: 71–75.

Appendices: Translations

In the translations my mathematical comments are given indented in the text, while philological comments are given in footnotes.

1. *Jiuzhang suanshu* 九章算術, Chapter 9, Problem 20, with Liu Hui's 劉徽 explanation

The text is taken from the critical edition by Guo Shuchun (2004: 419; note also 2009: 255–256). Differences from Qian Baocong's edition (1963: 255–256) are minimal and have no effect on the interpretation.

今有邑方不知大小，各中開門。出北門二十步有木。出南門一十四步，折而西行一千七百七十五步見木。問：邑方幾何？
荅日：二百五十步。
術日：以出北門步數乘西行步數，倍之，爲實。
【此以折而西行爲股，自木至邑南一十四步爲句，以出北門二十步爲句率，北門至西隅爲股率，半廣數。故以出北門乘折西行股，以股率乘句之冪。然此冪居半，以西行，故又倍之，合東，盡之也。】
幷出南、北門步數，爲從法。開方除之，卽邑方。
【此術之冪，東西如邑方，南北自木盡邑南十四步之冪。各南、北步爲廣，邑方為袤，故連兩廣爲從法。并以爲隅外之冪也。】

[***Jiuzhang suanshu*:**] A square city of unknown size has gates at the centre of each [side]. At [a =] 20 paces outside the northern gate there is a tree.

> See Figure 2. The tree is at A and the northern gate is at B.

If one goes [b =] 14 paces out the southern gate, turns and walks [c =] 1,775 paces west, one sees the tree.

> The southern gate is at C. One walks to D, turns, and walks to E.

How large is [x =] the side of the city?

Answer: 250 paces.

Method: Multiply [a =] the [number of] paces out of the northern gate by [c =] the number of paces walked west; double this; this is the *shi* [the constant term in a polynomial].

> *shi* = *ac*

[**Liu Hui:**] Here [c =] the distance walked west after turning is the height [*gu* 股] [of a right triangle], and [$a + b + x =$] the distance from the tree to 14 paces south of the city is the base [*gou* 勾].

> The triangle is *ADE*.

Take b =] the [distance] out of the northern gate, 20 paces, to be the proportion of the base [*gou lü* 勾率], and the [distance] from the northern gate to the western corner, that is, half of [x =] the breadth [of the city], to be the proportion of the height [*gu lü* 股率].

> These 'proportions' are the height and base of the triangle *ABL*, which is similar to *ADE*.

Therefore multiplying [a =] the northern-gate proportion of the base by [c =] the west-walked height gives an area which is the product of [$½x$ =] one-half of the breadth, that is, the proportion of the height, and [$a + b + x =$] the base.

> See Figure 2. Because the triangles *ABL* and *ADE* are similar,

$$\frac{a}{½x} = \frac{a + b + x}{c}$$

> and

$$ac = ½x(a + b + x)$$

But this area is [*FADJ*,] the western half [of the rectangle *FGHJ*]. Therefore it is further doubled to include the eastern half and complete it.

[***Jiuzhang suanshu*:**] Add together the numbers of paces from the northern and southern gates [b and a] to make the *zongfa* 從法 [the linear coefficient in a polynomial].

$$zongfa = a + b$$

The equation is then

$$x^2 + (a + b)x = ac$$

Extract the square root. This is the side of the city.

[**Liu Hui:**] The area in this calculation is like an area which from east to west is [the length of] the side of the city and from south to north is [the distance] from the tree through to 14 paces south of the city $[a + x + b]$. All of the paces south and north form the breadth $[a + b]$ [of the *zongfa*] and the side of the city is the length. The sum [of these two areas] is the area outside the corner [in the geometric explanation of the square-root algorithm].

Figure 3 shows how the geometric version of the equation is formed. Compare Figure 1b. The 'corner' is the small square *PQTR* and the two rectangles *KGQP* and *PRUS* together form the *zongfa*.

2. *Jigu suanjing* 緝古算經, by Wang Xiaotong 王孝通, second part of Problem 3

The text is taken from Qian Baocong's critical edition (1963: 526–527) with corrections by Guo Shuchun and Liu Dun (1998: 217). The translation is copied without significant change from Lim and Wagner (2017: 145–148).

In the text Wang Xiaotong gives names to two intermediate quantities in the calculation. I denote these with the symbols K_1 and K_2.

求甲縣高、廣、正、斜袤術曰：
以程功乘甲縣人，以六因取積。又乘袤冪，以下廣差乘高差，以法，除之，為實。又幷小頭上、下廣，以乘小高，三因之，為垣頭冪。又乘袤冪，如法而一，為垣方。又三因小頭下廣，以乘正袤，以廣差除之，為都廉，從。開立方除之，得小頭即甲袤。又以下廣差乘之，所得，以正袤除之。所得，加東頭下廣，即甲廣。又以兩頭高差乘甲袤，以正袤除之，以加東頭高，即甲高。又以甲袤自乘；以隄東頭高減甲高，餘自乘，幷二位，以開方除之，即得斜袤。若求乙、丙、丁，各以本縣人功積尺，每以前大高、廣為後小高、廣。凡廉母自乘為方母，廉母乘方母為實母。

In the following see Figure 4.

The method for calculating the height, the width, and the straight and slanted lengths of [the part of the dyke built by] county A is:

Multiply the regulation amount of work by the number of men in county A.

$$V_{\mathrm{A}} = 4{,}960\ cun^3/\ \text{man} \times 6{,}724\ \text{men} = 33{,}351{,}040\ cun^3$$

Multiply the volume obtained [V_A] by 6. Multiply this by the area of [the square on] the length [l]. Multiply the difference between the lower widths [b_2–b_1] by the difference between the heights [h_2–h_1] to make [K_1 =] the Divisor and divide to make the *shi* 實 [the constant term in the cubic equation].

$$K_1 = (b_2 - b_1)(h_2 - h_1) = 192{,}200\ cun^2$$

$$shi = \frac{6V_{\mathrm{A}}l^2}{K_1} = 23{,}987{,}761{,}548\tfrac{12}{31}\,cun^2$$

Further add together the upper and lower widths of the small end [a_1, b_1] and multiply by the small height [h_1]. Multiply this by three to make [K_2 =] the Area for the End Wall.

$$K_2 = 3(a_1 + b_1)h_1 = 20{,}646\ cun^2$$

Multiply by the area of [the square on] the length [l] and divide by the Divisor [K_1] to obtain the *yuanfang* 垣方 [the coefficient of the linear term].

$$yuanfang = \frac{K_2 l^2}{K_1} = 2{,}474{,}941\tfrac{29}{31}\,cun^2$$

Triple the lower width of the small end [b_1] and multiply by the straight length [l]. Divide by the difference in the widths [b_2–b_1] to make the *dulian* 都廉 [the coefficient of the quadratic term].

$$dulian = \frac{3b_1 l}{b_2 - b_1} = 3{,}298\tfrac{2}{31}\,cun$$

Extract the cube root to obtain the length at the small end [x], which is the length of [the contribution of] county A.

$$x^3 = \frac{3b_1 l}{b_2 - b_1}x^2 + \frac{K_2 l^2}{K_1}x = \frac{6V_{\mathrm{A}}l^2}{K_1}$$

$$x^3 + 3{,}298\tfrac{2}{31}x^2 + 2{,}474{,}941\tfrac{29}{31}x = 23{,}987{,}761{,}548\tfrac{12}{31}\,cun^2$$

$$x = 1{,}920\ cun$$

Further multiply this by the difference between the lower widths [b_2–b_1] and divide by the straight length [l]. To the result add the lower width of the eastern end [b_1]. This is the width [$b_{2\mathrm{A}}$] of [the contribution of] county A.

Multiply the difference between the heights of the two ends [h_2–h_1] by the length [x] of [the contribution of] county A. Divide by the straight length [l] and add the height of the eastern end [h_1]. This is the height [h_A] of [the contribution of] county A.

$$b_{2A} = \frac{x(b_2 - b_1)}{l} + b_1 = 390\ cun$$

$$h_A = \frac{(h_2 - h_1)x}{l} + h_1 = 155\ cun$$

Further multiply the length [x] of [the contribution of] county A by itself. Subtract the height [h_1] of the eastern end of the dyke from the height [h_A] of [the contribution of] county A. Multiply the difference by itself. Add the two quantities and extract the square root to obtain the slanted length [s_A].

$$s_A = \sqrt{x^2 + (h_A - h_1)^2} = 1{,}924\ cun$$

In calculating [the contributions of] counties B, C and D, use the number of men in the respective county and the regulation amount of manual work in [cubic] *chi* [that one man can do] [to obtain the volume]. For each [county] let the previous large height and width be the later small height and width.

In each case the denominator of the *fang*[*fa*] [the linear coefficient], multiplied by itself, is the denominator of the *lian*[*fa*] [the quadratic coefficient], and the denominator of the *fang*[*fa*] multiplied by the denominator of the *lian*[*fa*] is the denominator of the *shi* [the constant term].

> This sentence describes the normalization of the fractions involved in the cubic equation. See Lim and Wagner 2017: 29–30; Wagner 2017.

2.1. Comment in smaller characters

此平隄在上，羨除在下。兩高之差即除高。其除兩邊各一鼈腝，中一塹堵。今以袤再乘積，廣差乘高差而一，得截鼈腝袤再乘為立方一。又塹堵袤自乘為冪一，又三因小頭下廣，大袤乘之，廣差而一，與冪為高，故為廉法。又幷小頭上、下廣，又三之，以乘小頭高，為頭冪，意同六除。

然此頭冪本乘截袤。又袤再乘之，差相乘而一。今還依數乘除頭冪，為從。開立方除之，得截袤為廣。

> This is a description of the volume dissection in Figure 5:

In this case there is a level dyke [*DCKJHEFG*] at the top and a *yanchu* [*ABKJHG*] at the bottom. The difference between the two heights is then the height of the [*yan*]*chu*. The [*yan*]*chu* has on each side a *bienao* [*AJLH* and *BMKG*] and in the middle a *qiandu* [*JKMLHG*]. . . .

> The rest of the comment appears to be corrupt. We discuss it in detail in Lim and Wagner 2017: 148–151.

3. *Jigu suanjing*, Problems 19–20

The text is taken from Qian Baocong's critical edition (1963: 526–527), with the addition of Zhang Dunren's speculative reconstruction of missing parts of the text (1803: 131–133). The translation is copied without significant change from Lim and Wagner (2017: 202–203).

Problems 15–20 in the *Jigu suanjing* are three pairs of related problems, and it is the parallel between Problems 19 and 20 that permits reconstructions by Zhang Dunren and Nam Pyŏng-Gil (1985: 362–363). These two reconstructions are discussed in detail by Lim and Wagner (2017: 104–108). They use different numbers, but the calculations are the same.

In both the text and the translation, Zhang Dunren's additions are enclosed in braces { }.

3.1. Problem 19

假令有股弦相乘冪{七百二十六，句七、十分之}七。問股多少？
答曰：股二十{六、五分之二。}
術曰：冪自{乘，為實。句自乘，為

方法，從。開方}除之，所得，{又開方，即股。}

In the following consult Figure 6.

[In a right triangle] the area obtained by multiplying the leg [*b*] by the hypotenuse [*c*] is [*bc* =] {726, and the base is [*a* =] $7^{7}/_{10}$.} How large is the leg [*b*]?

Answer: The leg is [*b* =] {$26^{2}/_{5}$}.

Method: The area {is multiplied} by itself {to make the *shi* 實 [the constant term in the equation].

$$shi = (bc)^2$$

The base [*a*] is multiplied by itself to make the *fangfa* 方法 [the linear coefficient].

$$fangfa = a^2$$

Extract the square root.

$$x^2 + a^2x = (bc)^2$$

$$x = 696\frac{24}{25}$$

Further extract the square root of} the result. {This is the leg [*b*]}.

$$b = \sqrt{x} = 26\frac{2}{5}$$

3.2. Comment in smaller characters

□□□□□□□□□□□□□□□數亦是股□□□□□□□□□□□□為長以股
□□□□□□□□□□□□□得股冪又開。。。股北分母常。。。

. . . the number is also the leg [*b*] . . . to make the length [*chang* 長], . . . by the leg [*b*] . . .

3.3 Problem 20

假令有股十六、二分{之一，句弦相乘冪一百六}十四、二十五分{之十四。問句多少？}
答曰：{八、五分之四。}
術曰：冪自乘{為實。股自乘，為方法，從。開方}除之，所得，又開方{即句。}

[In a right triangle] the leg [b] is 16 and {one} half {and the base multiplied by the hypotenuse is [ac =] 16}$4^{24}/_{25}$. {How large is the base [a]?}

Answer: {$8^4/_5$}.

Method: The area is multiplied by itself {to make the *shi* [the constant term in the equation].

$$shi = (ac)^2$$

The base [a] is multiplied by itself to make the *fangfa* [the linear coefficient].}

$$fangfa = b^2$$

Extract {the square root}.

$$x^2 + b^2x = (ac)^2$$

$$x = 77^{11}/_{25}$$

Further extract the square root of the result. {This is the base [a]}

$$a = \sqrt{x} = 8^4/_5$$

4. *Ceyuan Haijing* 測圓海鏡, by Li Ye 李冶

The text is taken from the *Zibuzuzhai* 知不足齋 edition.[17] The translations owe much to Karine Chemla (1982). In all of the following see Figure 8.

4.1. Background: Excerpts from Chapter 1

4.1.1. Names of line segments and triangles

天之地為通弦，天之乾為通股，乾之地為通勾。
天之川為邊弦，天之西為邊股，西之川為邊勾。(1:1b, lines 2–5)

From *tian* [*C*] to *di* [*T*] is the *tongxian* [*CT*, the hypotenuse of the *tong* (‘full’) triangle, *CQT*]; from *tian* [*C*] to *qian* [*Q*] is the *tonggu* [*QC*, its height]; from *qian* [*Q*] to *di* [*T*] is the *tonggou* [*QT*, its base].

From *tian* [*C*] to *chuan* [*F*] is the *bianxian* [*CF*, the hypotenuse of the *bian* (‘side’) triangle, *CFO*]; from *tian* [*C*] to *xi* [*O*] is the *biangu* [*CO*, its height]; from *xi* [*O*] to *chuan* [*F*] is the *biangou* [*OF*, its base].

月之川為上平弦，月之青為股，青之川為勾。
川之地為下平弦，川之夕為股，夕之地為勾。
天之月為大差弦，天之坤為股，坤之月為勾。
山之地為小差弦，山之艮為股，艮之地為勾。 (1:2a, lines 8–9, 2b, 1–6)

From *yue* [*L*] to *chuan* [*F*] is the *shangpingxian* [*LF*, the hypotenuse of the *shangping* (‘upper even’) triangle, *LFV*]; from *yue* [*L*] to *qing* [*V*] is the [*shangping*]*gu* [*LV*, its height]; from *qing* [*V*] to *chuan* [*F*] is the [*shangping*]*gou* [*VF*, its base].

17 ctext.org/library.pl?res=3094.

From *chuan* [*F*] to *di* [*T*] is the *xiapingxian* [*FT*, the hypotenuse of the *xiaping* ('lower even') triangle, *FTZ*], from *chuan* [*F*] to *xi* [*Z*] is the [*xiaping*]*gu* [*FZ*, its height]; from *xi* [*Z*] to *di* [*T*] is the [*xiaping*]*gou* [*ZT*, its base].

Note that these two triangles, *LFV* and *FTZ*, are congruent.

From *tian* [*C*] to *yue* [*L*] is the *dachaxian* [*CL*, the hypotenuse of the *dacha* (large difference) triangle, *CLK*]; from *tian* [*C*] to *kun* [*K*] is the [*dacha*]*gu* [*CK*, its height], from *kun* [*K*] to *yue* [*L*] is the [*dacha*]*gou* [*KL*, its base].

From *shan* [*B*] to *di* [*T*] is the *xiaochaxian* [*BT*, the hypotenuse of the *xiaocha* ('small difference') triangle, *BTG*]; from *shan* [*B*] to *gen* [*G*] is the[*xiaocha*]*gu* [*BG*, its height]; from *gen* [*G*] to *di* [*T*] is the [*xiaocha*]*gou* [*TG*, its base].

4.1.2. Geometric identities

以邊弦減大股，餘為半徑內減平勾，又為平弦內減小差也。(1:18b, lines 2–3)

Subtracting the *bianxian* [*CF*] from the *dagu* ['large height', i.e. the *tonggu* (*QC*)], the difference is half the diameter [*d*] minus the [*xia*]*pinggou* [*ZT*], and also the [*xia*]*pingxian* [*FT*] minus the *xiaocha*[*gou*] [*TG*].

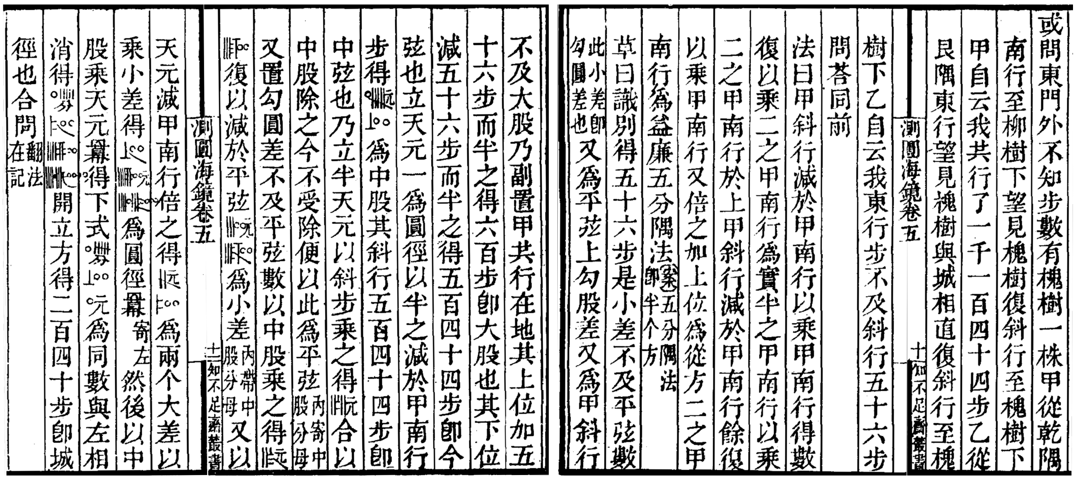
$$QC - CF = FT - TG = \frac{1}{2}d - ZT$$

凡大、小差相乘為半段徑冪【大差勾、小差股相乘亦同上】。 (1:9b, lines 4–5)

The *dacha*[*gu*] [*CK*] multiplied by the *xiaocha*[*gou*] [*TG*] is half the area of [the square on] the diameter [*d*]. The *dachagou* [*KL*] multiplied by the *xiaochagou* [*BG*] is the same.

$$CK \times TG = KL \times BG = \frac{1}{2}d^2$$

4.2. Chapter 5, Problem 12


或問東門外不知步數有槐樹一株甲從乾隅南行至柳樹下望見槐樹復斜行至槐樹下甲自云我共行了一千一百四十四步乙從艮隅東行望見槐樹與城相直復斜行至槐樹下乙自云我東行步不及斜行五十六步

問荅同前

法曰甲斜行減於甲南行以乘甲南行得數復以乘二之甲南行爲實半之甲南行以乘二之甲南行於上甲斜行減於甲南行餘復以乘甲南行又倍之加上位爲從方二之甲南行爲益廉五分隅法（案五分隅法卽半个方）

草曰識別得五十六步是小差不及平弦數（此小差卽勾圓差也）又爲平弦上勾股差又爲甲斜行不及大股乃副置甲共行在地其上位加五十六步而半之得六百步卽大股也其下位減五十六步而半之得五百四十四步卽今弦也立天元一爲圓徑以半之減於甲南行步得　爲中股其斜行五百四十四步卽中弦也乃立半天元以斜步乘之得　合以中股除之今不受除便以此爲平弦（內寄中股分母）又置勾圓差不及平弦數以中股乘之得　復以減於平弦　爲小差（內帶中股分母）又以天元減甲南行倍之得　爲兩个大差以乘小差得　爲圓徑冪（寄左）然後以中股乘天元冪得下式　爲同數與左相消得　開立方得二百四十步卽城徑也合問（在翻法記）

測圓海鏡卷五　十一　知不足齋叢書

測圓海鏡卷五　十二　知不足齋叢書

(5: 10a–11b)

At an unknown number of paces outside the eastern gate [*E*] is a pagoda tree [*F*]. Jia walks south from the *qian* 乾 corner [*Q*] to a willow tree [*C*] and sees the pagoda tree. He further walks obliquely to the pagoda tree [*F*]. Jia says to himself, ‘I have in all walked [*QC* + *CF* =] 1,144 paces.’

Yi walks east from the *gen* 艮 corner [*G*] [to *T*] and sees the pagoda tree [*F*] in line with the wall. He further walks obliquely to the pagoda tree [*F*]. Yi says to himself, ‘The paces that I walked east [*TG*] are less than those I walked obliquely [*FT*] by [*FT* – *TG* =] 56 paces.

The question and the answer are the same as before, [i.e., the diameter *d* of the circular wall is required; the answer is 240 paces].

$QC + CF = 1{,}144$ paces

$FT - TG = 56$ paces

$d = ?$

Method: Subtract the distance walked obliquely by Jia [*CF*] from the distance walked south by Jia [*QC*]; multiply this by the distance walked south by Jia [*QC*] to obtain a number.

This 'number', (*QC*–*CF*)×*QC*, will be used again below.[18]

Further multiply this by twice the distance walked south by Jia [*QC*]; this is the *shi* 實 [the constant term in a polynomial equation].

Elsewhere in the book (see above, 1:18b, lines 2–3) it is stated that *QC*–*CF* is equal to *FT*–*TG*. Therefore an easy calculation gives

$QC+CF$ = 1,144 paces

$QC-CF = FT-TG$ = 56 paces

QC = 600 paces

CF = 544 paces

This calculation is made explicit in the 'Details' section below. It is not clear why it is not explicit here. It follows that

$$shi = (QC-CF) \times QC \times 2QC$$

$$= (600-544) \times 600 \times 2 \times 600 = 40{,}320{,}000 \text{ cubic paces}$$

Multiply half the distance walked south by Jia [*QC*] by twice the distance walked south by Jia [*QC*] and place this at the *shang* 上 [position on the calculating board].

$$shang \text{ position} = \tfrac{1}{2}QC \times 2QC = 360{,}000 \text{ square paces}$$

Subtract the distance walked obliquely by Jia [*CF*] from the distance walked south by Jia [*QC*]. Further, multiply this difference by the distance walked south by Jia [*QC*] and double it.

---

[18] See fn. 19 below.

This, $(QC–CF) \times QC \times 2$, is twice the 'number' mentioned above.[19]

Add this to the *shang* position. This is the *zongfang* 從方 [the coefficient of the linear term].

$$zongfang = shang + (QC - CF) \times CQ \times 2$$

$$= \frac{1}{2}QC \times 2QC + (QC - CF) \times 2 = 427{,}200\ paces$$

Twice the distance walked south by Jia [*QC*] is [minus] the *yilian* 益廉 [the coefficient of the quadratic term].

$$yilian = -2QC = -1{,}200 \text{ paces}$$

Five tenths is the *yufa* 隅法 [the coefficient of the cubic term].[20]

These calculations result in the equation

$$0.5\boldsymbol{d}^3 + 2QC\boldsymbol{d}^2 + \left[\frac{1}{2}QC \times 2QC + (QC - CF) \times QC \times 2\right]\boldsymbol{d}$$

$$= (QC - CF) \times QC \times 2QC$$

$$0.5\boldsymbol{d}^3 - 1{,}200\boldsymbol{d}^2 + 427{,}200\boldsymbol{d} = 40{,}320{,}000 \text{ cubic paces} \qquad (22)$$

In which *d* is the diameter of the circular city. This equation has three roots, *d* = 168.737, 240, or 1,991.26 paces. The first two are correct answers to the problem, but only the second, 240, is noted in the text.

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[19] See fn. 18 above.

[20] Obscure comment in smaller characters omitted in the translation.

Details: From the statement of facts [*shibie* 識別][21] is obtained that the 56 paces, the difference between the *xiaocha* 小差 [*TG*] and the *pingxian* 平弦 [*FT*], (this *xiaocha* is the same as the *gouyuancha* 勾圓差)[22] is also the difference between the height [*FZ*] and the [*TZ*] base of the *ping* 平 [*FTZ*] triangle and also the difference between the distance walked obliquely by Jia [*CF*] and the *dagu* 大股 [*QC*].

$$QC–CF = FZ–TZ = FT–TG = 56 \text{ paces}$$

So, as an adjunct [*fu* 副], the total distance walked by Jia is placed at *di* 地 [on the counting board]. In the upper position [of *di*], 56 paces is added and [the sum is] halved, obtaining 600 paces; this is the *dagu* 大股 [*QC*]. In the lower position, 56 paces is subtracted and [the sum is] halved, obtaining 544 paces; this is the *jinxian* 今弦 [*CF*].

$$QC+CF = 1{,}144 \text{ paces}$$

$$QC–CF = 56 \text{ paces}$$

$$QC = \frac{1}{2}(1{,}144 + 56)\ 600 \text{ paces}$$

$$CF = \frac{1}{2}(1{,}144 - 56) = 544 \text{ paces}$$

Letting the *tianyuanyi* 天元一 [the variable in a polynomial] be [*d*=] the diameter of the circle, halving it, and subtracting it from the distance walked south by Jia [*QC*] gives

[$CO = –0.5d + QC = –0.5d + 600$]

which is the *zhonggu* 中股 [*CO*]. The distance walked by him obliquely, 544 paces, is the *zhongxian* 中弦 [*CF*].

21 See Section 4.1.2 above.

22 Comment in smaller characters.

Setting up half of the *tianyuan* [*d*] and multiplying it by the oblique paces [*CF* = 544 paces] gives

[$0.5d \times CF = 272d$]

This is divided by the *zhonggu* 中股 [*CO*]. At this point it is not to be divided. Let this be the *pingxian* 平弦 [*FT*].

$$FT = \frac{0.5d \times CF}{CO} = \frac{0.5d \times 544}{600 - 0.5d} = \frac{272d}{600 - 0.5d}$$

(Note internally that the *zhonggu* [*CF* = 600–0.5*d*] is to be the denominator of a fraction.)[23]

Further setting up the difference between the *gouyuancha* 勾圓差 [*TG*] and the value of the *pingxian* 平弦 [*FT*] and multiplying by the *zhonggu* 中股 [*CO*] gives

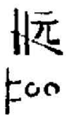
[$(FT–TG)\times CO = 56 \times (–0.5d + 600) = –28d + 33{,}600$]

Again subtracting from the *pingxian* 平弦 [*FT*]:

$$\left[FT - (FT - TG) = \frac{272d - (-28d + 33{,}600)}{600 - 0.5d} = \frac{300d - 33{,}600}{600 - 0.5d}\right]$$

This is the *xiaocha* 小差 [*CT*–*QC*]. (Internally this has the *zhonggu* [*CO* = 600–0.5*d*] as a denominator.)[24]

Further, subtracting the *tianyuan* 天元 [*d*] from the distance walked by Jia south and doubling gives



[$2 \times CK = –2d + 2 \times QC = –2d + 1{,}200$]

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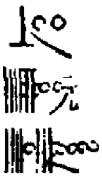
23 Comment in smaller characters.

24 Comment in smaller characters.

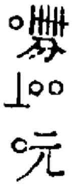
which is twice the *dacha* 大差 [2×*CK*]. Multiplying by the *xiaocha* 小差 [*CT*–*QC*] gives



$$\left[\frac{-QC^2 \times d^2 + (QC^2 + 2 \times 56 \times QC)d - 2 \times 56 \times QC^2}{QC - ½d} = \frac{-600d^2 + 427{,}200d - 40{,}320{,}000}{600 - 0.5d}\right]$$

This is the area of [the square on] the diameter of the circle [d^2]. (This is moved to the left.)[25]

Then the area of [the square on] the *tianyuan* 天元 [d^2] is multiplied by the *zhonggu* 中股 [*CO*], obtaining this arrangement:



$[CO \times d^2 = -0.5d^3 + 600d^2]$

This is the same number [ignoring the 'internal' denominator]. Simplifying with what is on the left gives[26]

$$\left[\begin{array}{c} 0.5d^3 - 2 \times QC \times d^2 + QC\,(2 \times 56 + QC)d - 2 \times 56 \times QC^2 \\ = 0.5d^3 - 1{,}200d^2 + 427{,}200d - 40{,}320{,}000 = 0 \end{array}\right] \quad (23)$$

Extracting the cube root gives 240 paces. This is the diameter of the city, in accordance with what was asked.[27]

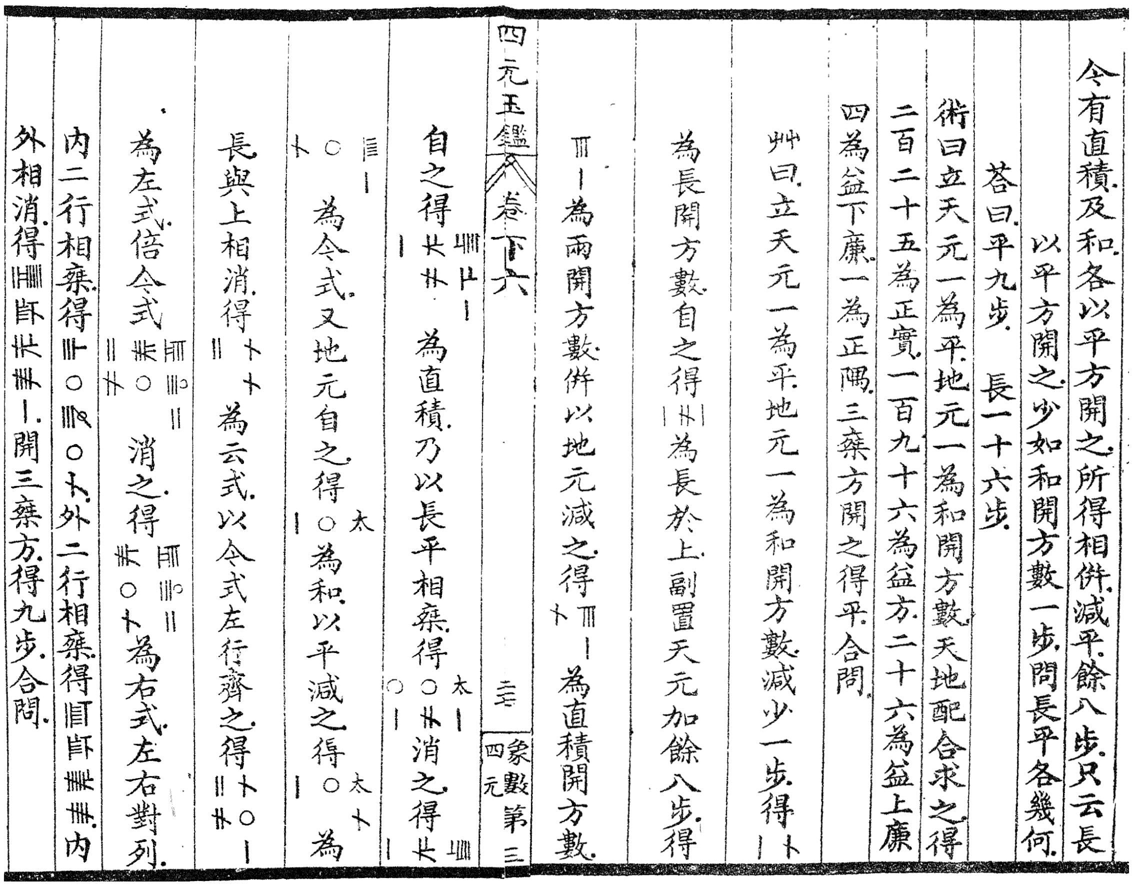
25 Comment in smaller characters.

26 Note the difference between (22) and (23).

27 Obscure comment in smaller characters omitted in the translation.

5. *Siyuan yujian xicao* 四元玉鑑細草, by Zhu Shijie 朱世傑 with detailed working by Luo Shilin 羅士琳, Chapter *xia* 下, Section 6, Problem 15, pp. 27a-b

The text is taken from the *Gujin suanxue congshu* 古今算學叢書 edition, 1897.[28]



今有直積及和各以平方開之所得相併減平餘八步只云長
以平方開之少如和開方數一步問長平各幾何
荅曰平九步 長一十六步
術曰立天元一為平地元一為和開方數天地配合求之得
二百二十五為正實一百九十六為益方二十六為益上廉
四為益下廉一為正隅三乘方開之得平合問
艸曰立天元一為平地元一為和開方數減少一步得 [illegible]
為長開方數自之得 [illegible] 為長於上副置天元加餘八步得
[illegible] 為兩開方數併以地元減之得 [illegible] 為直積開方數
四元玉鑑 卷下六 四元 象數第三
自之得 [illegible] 為直積乃以長平相乘得 [illegible] 消之得 [illegible]
[illegible] 為今式又地元自之得 [illegible] 為和以平減之得 [illegible] 為
長與上相消得 [illegible] 為云式以今式左行齊之得 [illegible]
為左式倍今式 [illegible] 消之得 [illegible] 為右式左右對列
內二行相乘得 [illegible] 外二行相乘得 [illegible] 內
外相消得 [illegible] 開三乘方得九步合問

[**Problem**, by Zhu Shijie:]

[28] https://commons.wikimedia.org/wiki/Category:%E5%8F%A4%E4%BB%8A%E7%AE%97%E5%AD%B8%E5%8F%A2%E6%9B%B8). Cf. Guo Shuchun et al. 2013, 2: 642–643.

(*Jin you* 今有) [In a rectangle with width a and length b], if the square roots of the area [*ab*] and the sum [*a*+*b*] are extracted, the results are added together, and the width [*a*] is subtracted, the difference is 8 paces.

$$\sqrt{ab} + \sqrt{a+b} - a = 8 \text{ paces}$$

(*Zhi yun* 只云) If the square root of the length [*b*] is extracted, it is less than the square root of the sum [*a* + *b*] by 1 pace.

$$\sqrt{a+b} - \sqrt{b} = 1 \text{ pace}$$

How large are the length [*b*] and the width [*a*]?

Answer: The width is [*a* =] 9 paces and the length is [*b* =] 16 paces.

Method: Let the *tianyuanyi* 天元一 be the width [*x* = *a*] and the *diyuanyi* 地元一 be the square root of the sum $\left[y = \sqrt{a+b}\right]$. Calculate by combining the *tian* and the *di* [*x* and *y*].

The result is [a quartic equation with] 225 as positive *shi* 實 [constant term], 196 as negative *fang* 方 [coefficient of the linear term], 26 as negative *shanglian* 上廉 [coefficient of the quadratic term], 4 as negative *xialian* 下廉 [coefficient of the cubic term], and 1 as positive *yu* 隅 [coefficient of the quartic term].

This equation is

$$x^4 - 4x^3 - 26x^2 - 196x + 225 = 0$$

Extract the quartic root to obtain the width [*x* = *a*], in accordance with what was asked.

The equation has two real roots, 1 and 9. Only 9 is a correct solution to the problem.

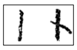
[**Detailed calculation**, by Luo Shilin:] Let the ***tianyuanyi*** [*x*] be the width [*a*].

$x = a$

Let the ***diyuanyi*** [y] be the square root of the sum.

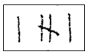
$$y = \sqrt{a+b}$$

Subtract 1 pace to obtain



$$\begin{vmatrix} 1 & -1 \end{vmatrix} \qquad [y-1]$$

which is the square root of the length $[\sqrt{b}\,]$. [Multiply] this by itself to obtain



$$\begin{vmatrix} 1 & -2 & 1 \end{vmatrix} \qquad [y^2-2y+1]$$

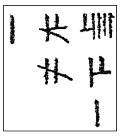
which is the length [b]. [Place this] at the top. Separately set up the *tianyuan* [x] plus the difference, 8 paces, obtaining

$$\begin{vmatrix} 8 \\ 1 \end{vmatrix} \qquad [x+8]$$

which is the sum of the two square roots $[\sqrt{ab}+\sqrt{a+b}]$. Subtract the *diyuan* $[y=\sqrt{a+b}\,]$ to obtain

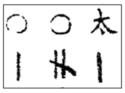
$$\begin{vmatrix} -1 & 8 \\ & 1 \end{vmatrix} \qquad [x-y+8]$$

which is the square root of the area $[\sqrt{ab}]$. [Multiply] this by itself to obtain



$$\begin{vmatrix} 1 & -16 & 64 \\ & -2 & 16 \\ & & 1 \end{vmatrix} \qquad [x^2+16x-2xy-16y+y^2+64]$$

which is the area $[ab]$. Then multiply the length [b, the expression saved above] by the width [$x = a$] to obtain

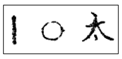


$$\begin{vmatrix} 0 & 0 & \text{太} \\ 1 & -2 & 1 \end{vmatrix} \qquad [x-2xy+xy^2 = ab]$$

and clear [the area, ab, calculated immediately above] with this to obtain

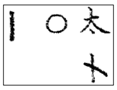
$$\begin{vmatrix} 1 & -16 & 64 \\ -1 & 0 & 15 \\ & & 1 \end{vmatrix} \qquad [x^2 + 15x - xy^2 - 16y + y^2 + 64 = 0]$$

This is the ***jin* array**. Further [multiply] the *diyuan* [*y*] by itself to obtain



$$\begin{vmatrix} 1 & 0 & \text{太} \end{vmatrix} \qquad [y^2]$$

which is the sum [*a*+*b*]. Subtract the width [*x* = *a*] to obtain

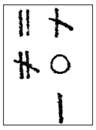


$$\begin{vmatrix} 1 & 0 & \text{太} \\ & & -1 \end{vmatrix} \qquad [-x + y^2]$$

which is the length [*b*]. Clear this with [the array saved at] the top $[y^2 - 2y + 1 = b]$ to obtain

$$\begin{vmatrix} 2 & -1 \\ & -1 \end{vmatrix} \qquad [-x + 2y - 1 = 0]$$

This is the ***yun* array**. Homogenize [*qi* 齊] with the left column of the *jin* array to obtain

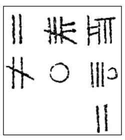


$$\begin{vmatrix} 2 & -1 \\ -2 & 0 \\ & 1 \end{vmatrix} \qquad [x^2 - 2xy + 2y - 1 = 0]$$

making the **left array**.

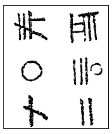
> The left column of the *jin* array is
>
> $$\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \; [-xy + y]$$
>
> This is divided by *y* and multiplied by the *yun* array to make the left array.

Double the *jin* array



$$\begin{vmatrix} 2 & -32 & 128 \\ -2 & 0 & 30 \\ & & 2 \end{vmatrix} \qquad [2x^2 + 30x - 2xy^2 - 32y + 2y^2 + 128 = 0]$$

and [with this] clear [the left array] to obtain



$$\left|\begin{array}{rr} -31 & 128 \\ 0 & 30 \\ -1 & 2 \end{array}\right| \qquad [2x^2 - x^2y + 30x - 31y + 128 = 0]$$

making the **right array**. Place the left and right [arrays] facing each other.

The board then looks like this:

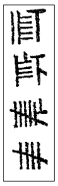
$$\left|\begin{array}{rr} 2 & -1 \\ -2 & 0 \\ & 1 \end{array}\right| \qquad \left|\begin{array}{rr} -31 & 128 \\ 0 & 30 \\ -1 & 2 \end{array}\right|$$

$$[x^2 - 2xy + 2y - 1 = 0] \qquad [2x^2 - x^2y + 30x - 31y + 128 = 0]$$

Multiplying the two inner columns [$x^2 - 1$ and $-x^2y - 31y$] results in

$$\left|\begin{array}{r} 31 \\ 0 \\ -30 \\ 0 \\ -1 \end{array}\right| \qquad \begin{bmatrix} (x^2 - 1) \times (-x^2y - 31y)/y \\ = \\ -x^4 - 30x^2 + 31 \end{bmatrix}$$

Multiplying the two outer columns [$-2xy + 2y$ and $2x^2 + 30x + 128$] results in



$$\left|\begin{array}{r} 256 \\ -196 \\ -56 \\ -4 \end{array}\right| \qquad \begin{bmatrix} (-2xy + 2y)/y \times (2x^2 + 30x + 128) \\ = \\ -4x^3 - 56x^2 - 196x + 256 \end{bmatrix}$$

Clearing the inner and outer [columns] results in

$$\left|\begin{array}{r} 225 \\ -196 \\ -26 \\ -4 \\ 1 \end{array}\right| \qquad \begin{bmatrix} [-4x^3 - 56x^2 - 196x + 256] - [-x^4 - 30x^2 + 31] \\ = \\ x^4 - 4x^3 - 26x^2 - 196x + 225 = 0 \end{bmatrix}$$

Extracting the fourth root, the result is [$a = x =$] 9 paces, in accordance with what was asked.

References

Chemla, Karine. 1982. *Étude du livre «Reflets des mesures du cercle sur la mer»*. Dissertation, University of Paris XIII.

Chemla, Karine, and Guo Shuchun. 2004. *Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Du Shiran 杜石然. 1966. 'Zhu Shijie yanjiu 朱世杰研究'. In *Song Yuan shuxue shi lunwenji* 宋元数学史论文集, ed. by Qian Baocong 钱宝琮. Beijing: Kexue Chubanshe, pp. 166–209.

Guo Shuchun 郭書春, ed. 2004. *Huijiao Jiuzhang suanshu* 匯校九章筭術（增補版）2nd ed. 2 vols. Shenyang: Liaoning Jiaoyu Chubanshe.

———. 2009. 九章筭术译注. Shanghai: Shanghai Guji Chubanshe.

Guo Shuchun 郭书春, Chen Zaixin 陈在新, and Guo Jinhai 郭金海. 2006. *Jade mirror of the four unknowns*. 2 vols. (Library of Chinese classics, Chinese–English / Da Zhonghua wenku, Han–Ying duizhao 大中华文库，汉英对照). Shenyang: Liaoning Education Press / Liaoning Jiaoyu Chubanshe. *Siyuan yujian* 四元玉鍵 (1303 CE), by Zhu Shijie 朱世杰. Full text + modern Chinese translation by Guo Shuchun and English translation by Chen Zaixin. Revised and supplemented by Guo Jinhai.

Guo Shuchun 郭书春, Joseph W. Dauben, and Xu Yibao 徐义保. 2013. *Nine chapters on the art of mathematics*. 3 vols. Shenyang: Liaoning Education Press. 'A critical edition and English translation based upon a new collation of the ancient text and modern Chinese translation by Guo Shuchun; English critical edition and translation, with notes by Joseph W. Dauben and Xu Yibao'.

Guo Shuchun 郭书春, and Liu Dun 刘钝, eds. 1998. *Suanjing shi shu* 算经十书. 2 vols. Shenyang: Jiaoyu Chubanshe.

Hoe, Jock. 1978. 'The Jade mirror of four unknowns – some reflections'. *Mathematical chronicle* 7: 125–156.

———. 2007. *The jade mirror of the four unknowns by Zhu Shijie: an early fourteenth century mathematics manual for teaching the derivation of systems of polynomial equations in up to four unknowns: A study*. Christchurch, New Zealand: Mingming Bookroom.

Hoe, John [i.e. Jock]. 1977. *Les systèmes d'équations polynômes dans le Siyuan yujian (1303)*. (Mémoires de l'Institut des Hautes Études Chinoises, 6). Paris: Collège de France.

Hummel, Arthur W. 1943. *Eminent Chinese of the Ch'ing period (1644–1912)*. 2 vols., Washington, D.C.: Government Printing Office. Repr. in 1 vol., Taipei: Ch'eng Wen, 1970.

Lim, Tina Su Lyn [林淑铃], and Donald B. Wagner. 2017. *The continuation of ancient mathematics: Wang Xiaotong's Jigu suanjing, algebra and geometry in seventh-century China*. (NIAS reports, 51). Copenhagen: NIAS Press.

Martzloff, Jean-Claude. 1997. *A history of Chinese mathematics*. Translated by S. S. Wilson. Berlin / Heidelberg: Springer-Verlag. 'Corrected second printing', 2006. Orig. *Histoire des mathémathiques chinoises*, Paris: Masson, 1987.

Mei Rongzhao 梅荣照. 1966. 'Li Ye ji qi shuxue zhuzuo 李冶及其数学著作'. In *Song Yuan shuxue shi lunwenji* 宋元数学史论文集, edited by Qian Baocong 钱宝琮. Beijing: Kexue Chubanshe, pp. 104–148.

Nam Pyŏng-Gil 南秉吉 (1820–1869). 1985. 緝古演段. (韓國科學技術史資料大系). www.scribd.com/doc/27920990

Qian Baocong 錢寶琮, ed. 1963. *Suanjing shi shu* 算經十書. Beijing: Zhonghua Shuju.

Shen Kangshen 沈康身, John N. Crossley, and Anthony W.-C. Lun. 1999. *The nine chapters on the mathematical art: Companion and commentary*. Oxford and Beijing: Oxford University Press and Science Press.

Vanhée, L. 1931. 'Le Précieux miroir des quatre éléments'. *Asia Major* 78: 242–270.

Wagner, Donald B. 2013. 'Mathematics, astronomy, and the planning of public works in China, Han to Yuan'. donwagner.dk/SAW/SAW.html

———. 2017. 'The classical Chinese version of Horner's method: Technical considerations'. donwagner.dk/horner/horner.html

———. 2022. 'A geometric challenge from ancient China'. donwagner.dk/challenge/challenge.html

———. 2023. ' "Incorrect corrections" by ancient editors – a challenge in Chinese mathematical philology'. In *Tan shi qiu xin: Qingzhu Guo Shuchun xiansheng bashi huadan wenji* 探史求新：庆祝郭书春先生八十华诞文集, edited by Zou Dahai 邹大海, Guo Jinhai 郭金海, and Tian Sen 田森. Harbin: Harbin Gongye Daxue Chubanshe, pp. 70–95.

Yang Bojun 楊伯峻, ed. 1960. *Mengzi yizhu* 孟子譯注. 2 vols. Beijing: Zhonghua Shuju.

Zou Dahai 邹大海. 2022. *Zhongguo shuxue zai dianji shiqi de xingtai, chuangzao yu fazhan: Yi ruogan dianxing anli wei zhongxin de yanjiu* 中国数学在奠基时期的形态、创造与发展：以若干典型案例为中心的研究. Guangzhou: Guangdong Renmin Chubanshe.

Figure captions

Figure 1. The geometric interpretation of the square-root algorithm in the *Nine chapters*. **a**: The given number; **b**: after determining the first digit of the square root, 2; **c**: after the second digit, 3; **d**: after the third and last digit, 5. The result is 235 paces.

Figure 2. To illustrate Problem 20 of Chapter 9 in the *Nine chapters*.

Figure 3. The result of dissecting and rearranging parts of Figure 2.

Figure 4. To illustrate the second part of Problem 3 of Wang Xiaotong's *Continuation*.

Figure 5. Dissection of Figure 4 to derive the contribution of County A.

Figure 6. A right triangle, with the classical Chinese names for its parts and their commonest English translations.

Figure 7. Proposed construction to derive the leg of a right triangle in Problem 19 of Wang Xiaotong's *Continuation*.

Figure 8. **Left**: The diagram at the start of Li Ye's *Sea mirror*, with translations added for the characters used to mark points. **Right**: a representation of this diagram using letters to mark points. (For the convenience of readers the letters used here are the same as those used by Chemla (1982) in her version of the diagram.)

Figure 9. The two series of counting-rod numerals. Numerals of the two series are used alternately to allow closer spacing on the counting board.

Figure 10. Two examples of the use of counting rods (*chou* 籌) to represent polynomials.